Last time:

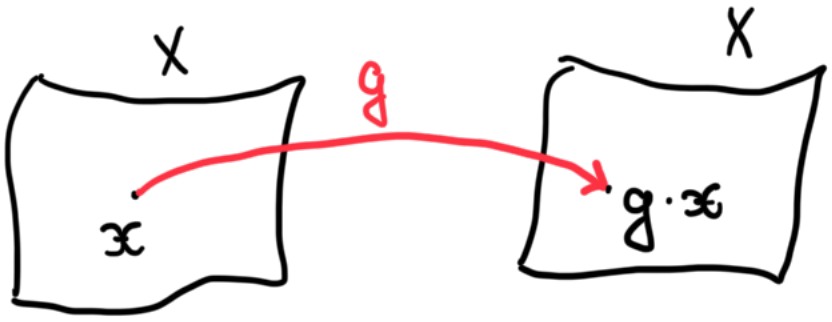
$G \curvearrowright X$

group

set

$\forall g \in G$, $\forall x \in X$ $\rightsquigarrow g \cdot x \in X$

a bunch of axioms



$\Longleftrightarrow$

homomorphism

{bijections X→X}

$G \xrightarrow{\Phi} S_X$

<u>Thm</u> (Cayley): any group G is isomorphic to a permutation group, i.e. a subgroup of S_X for some X

<u>Def</u>: let G be a group; a subset $H \subseteq G$ is called a subgroup ($H \leq G$) if H is closed under

- identity element, $e \in H$
- taking inverse, $\forall g \in H$, $g^{-1} \in H$
- operation, $\forall g, g' \in H$, $gg' \in H$

e.g.

$$\underset{\{\text{rot}\}}{\overset{}{\mathbb{Z}/n\mathbb{Z}}} \leq \underset{\{\text{rot}, \text{refl}\}}{D_{2n}}$$

$i : H \hookrightarrow G$ inclusion is a homomorphism

$f: G \longrightarrow G$ homomorphism; if f is injective, then

f induces an isomorphism $G \cong \operatorname{Im} f$

Def: if $f: G \longrightarrow G'$ is any homomorphism,

$$\operatorname{Ker} f = f^{-1}(e') = \{g \in G \mid f(g) = e'\}$$

Prop:
- $\operatorname{Ker} f$ is a subgroup of G and
- f is injective $\Longleftrightarrow$ $\operatorname{Ker} f$ is trivial
- $\operatorname{Im} f$ is a subgroup of G'

i.e. $= \{e\}$

Proof of Cayley: it suffices to construct an injective homomorphism $G \xhookrightarrow{\Phi} S_X$ for some X to be chosen

$\Updownarrow$

action $G \curvearrowright X$

Question: $G \curvearrowright X \rightsquigarrow \Phi: G \rightarrow S_X$

what is the meaning of $\operatorname{Ker} \Phi = \{g \in G \mid \Phi(g) = e\}$

$= \{g \in G \mid g \text{ acts on } X \text{ via } \mathrm{Id}_X\}$

$= \{g \in G \mid g \cdot x = x, \forall x \in X\}$

> **Def**: an action is called **faithful** if $\operatorname{Ker} = \{e\}$

$\Updownarrow$

any $e \neq g \in G$ actually moves X around, i.e. acts by a function different from Id_X

Ex: $D_{2n} \curvearrowright \{n\text{-gon}\}$ are faithful, but $G \curvearrowright X$ is not faithful

$S_n \curvearrowright \{1,\dots,n\}$

g [illegible] Id_X
i.e $g\cdot x = x$

To prove Cayley, it suffices to construct a faithful action of G on some set X: left action: $X = G$
$g\cdot h = gh$

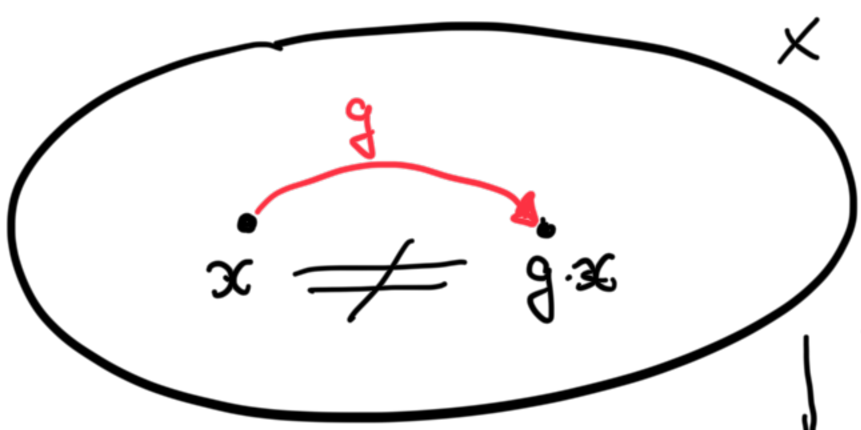
can $e \neq g$ act by Id_G on itself via left action?

$\Updownarrow$

contradict

$gh = h \quad \forall h \in G$ $\square$

Def: $G \curvearrowright X$ is called free if $g\cdot x \neq x$, $\forall e \neq g \in G$, $\forall x \in X$



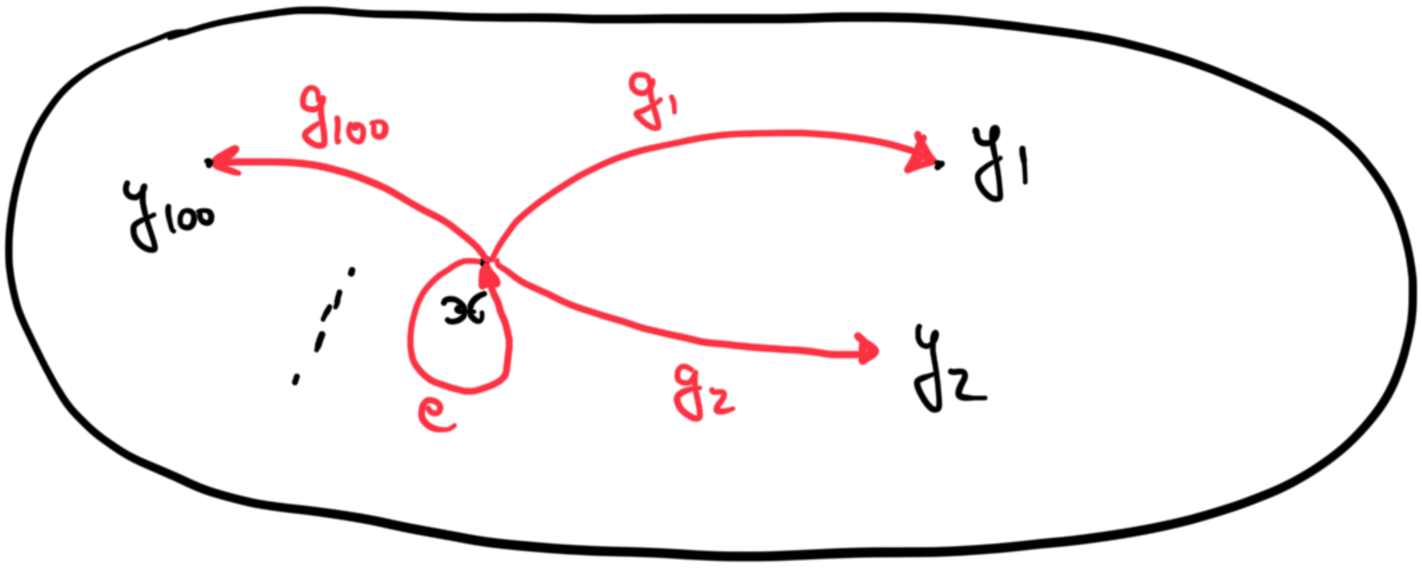
Ex: $D_{2n} \curvearrowright \{n\text{-gon}\} \curvearrowleft \mathbb{Z}/n\mathbb{Z}$ (rotations)
not free (for D_{2n}), free (for $\mathbb{Z}/n\mathbb{Z}$)

while the left action $G \curvearrowright G$ is free, the adjoint action $g\cdot h = ghg^{-1}$ need not be free

Def: given an action $G \curvearrowright X$, the following is an equivalence relation on X

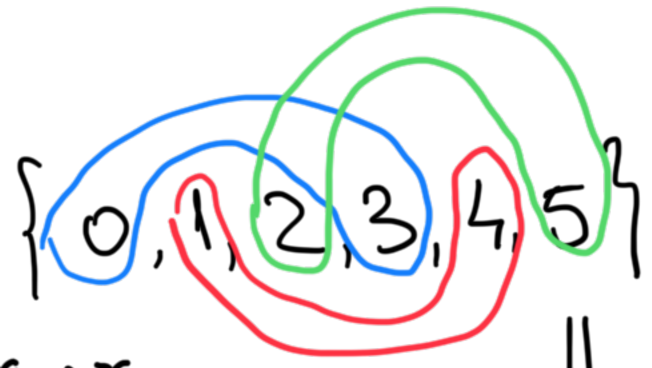
$$x \sim y \text{ if } \exists g \in G \text{ s.t. } g\cdot x = y$$

The equivalence classes of above equivalence relation are called orbits



$\{x, y_1, y_2, \ldots\ldots y_{100}, \ldots\ldots\}$ is the $\underbrace{\text{orbit of } x}_{\text{denoted } G\cdot x}$,

$$X = \bigsqcup_{\text{orbits } G\cdot x} G\cdot x$$



Example: $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{Z}/6\mathbb{Z} = \{0,1,2,3,4,5\}$

$= \{0,3\} \sqcup \{1,4\} \sqcup \{2,5\}$

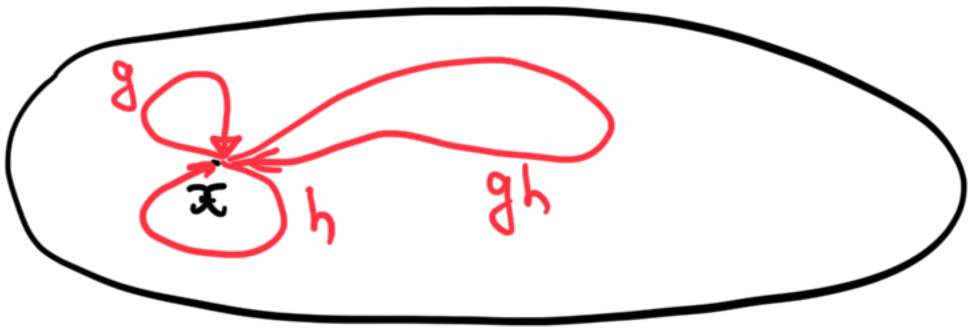
$0 \rightsquigarrow (+0)$, i.e. $x \to x$

$1 \rightsquigarrow (+3)$, i.e. $x \to 3+x$

Def: an action is transitive if $\exists$ a single orbit

Def: $G \curvearrowright X$, $\forall x\in X$, the stabilizer of x is

$$\mathrm{Stab}_G(x) = \{g\in G \mid g\cdot x = x\} \leqslant G$$



$g\cdot x = x$

$h\cdot x = x$ $\Rightarrow gh\cdot x = x$

Prop: if $x \sim y$ then $\mathrm{Stab}_G(x) \cong \mathrm{Stab}_G(y)$

$\Downarrow$ have the same size

$\Downarrow$

$\exists\, h \in G$ s.t. $\boxed{y = h \cdot x}$

$$\begin{tikzcd}
K = \mathrm{Stab}_G(x) \arrow[d, "F"', "g \rightsquigarrow hgh^{-1}"] & \leq G \arrow[d, "F"] \\
L = \mathrm{Stab}_G(y) & \leq G
\end{tikzcd}$$

Claim: $F : G \to G$, $F(g) = hgh^{-1}$ sends K to L

Proof: assume $g \in K$, i.e. $\boxed{g \cdot x = x}$

$$\left.\begin{aligned} hgh^{-1}y &= hg\cancel{h^{-1}h}\cdot x \\ &= hg\cdot x = h\cdot x = y \end{aligned}\right\} \Longrightarrow hgh^{-1} \in L$$

Thm (Orbit-Stabilizer theorem)

if G is finite, $\forall\, x \in X$

$$\boxed{|G \cdot x| = \frac{|G|}{|\mathrm{Stab}_G(x)|}}$$

size of orbit

size of stabilizer

Proof: fix $x \in X$; consider function

$$f : G \longrightarrow X$$
$$g \rightsquigarrow g \cdot x$$

① $\mathrm{Im}\, f = G \cdot x$
$= \{ g \cdot x \mid g \in G \}$

② domain $f = G$

③ all fibers of f: $f^{-1}(x) = \{ g \in G \mid g \cdot x = x \} = \mathrm{Stab}_G(x)$

$\forall h\in G \quad f^{-1}(h\cdot x) = \{g\in G \mid g\cdot x = h\cdot x\} =$

have size $|\mathrm{Stab}_G(x)|$

$= \{g\in G \mid h^{-1}g\cdot x = x\} =$

$= \{g\in G \mid h^{-1}g \in \mathrm{Stab}_G(x)\}$

$= h\cdot \mathrm{Stab}_G(x)$

same size

①,②,③ and basic combinatorics $\Longrightarrow$ O-S theorem.

Burnside's Lemma: $G \curvearrowright X$

finite group finite set

fixed point set $\forall g\in G$ $\quad X^g := \{x\in X \mid g\cdot x = x\}$

$$\boxed{\#\text{orbits} = \sum_{g\in G} \frac{|X^g|}{|G|}}$$.

Ex: $\mathbb{Z} \curvearrowright \mathbb{R}$ $\quad$ orbits $= \left\{\begin{matrix}\text{real numbers with} \\ \text{given fractional part}\end{matrix}\right\}$

$k \mapsto (+k)$

<u>Def</u>: $H \leq G$; left action $H \curvearrowright G$

$h \mapsto$ sends g to hg

① orbits $= \{hg \mid h \text{ runs over } H\}$ for any g

$= H$ the right cosets

$= Hg$ are the right cosets

(right action $H \curvearrowright G$ has orbits = left cosets)
$h \rightsquigarrow$ sends g to gh^{-1}

② stabilizers of left action are trivial $\Longleftrightarrow$
$\Longleftrightarrow$ left action is free
($G \curvearrowright X$ is free $\Longleftrightarrow$ all stabilizers are trivial)

$$G = \bigsqcup_{\substack{\text{right}\\ \text{cosets}}} \underbrace{Hg}_{\text{orbit}} \Longrightarrow |G| = \sum_{\substack{\text{right}\\ \text{cosets}}} \underbrace{|Hg|}_{= \frac{|H|}{|\text{stab}|} = |H|} = \sum_{\text{right cosets}} |H|$$

$$\Longrightarrow \#\text{ of right cosets} = \frac{|G|}{|H|} \in \mathbb{Z} \Longrightarrow |H| \,\Big|\, |G|$$

(Lagrange)

<u>Notation</u>: G/H = set of left cosets
$H\backslash G$ = set of right cosets

$$\text{Above: } |H\backslash G| = |G/H| \underset{\text{if } |G|<\infty}{=} \frac{|G|}{|H|} = \underset{\text{index}}{[G:H]}$$

Orbits of the adjoint action: $G \curvearrowright G$

$g \cdot h = ghg^{-1}$

$$\|$$

$\{ghg^{-1} \mid g \in G\}$ for any given $h \in G$

$$\|$$

Conjugacy class of h

centralizer

$$\mathrm{Stab}_G(h) = \{ g \in G \mid gh = hg \} = C_G(h)$$

assume G is finite,

$$G = \bigsqcup_{\substack{\text{conjugacy}\\ \text{classes}}} \begin{smallmatrix}\text{conjugacy}\\ \text{class}\end{smallmatrix} \implies |G| = \sum_{\substack{\text{conjugacy}\\ \text{classes}}} \left| \begin{smallmatrix}\text{conjugacy}\\ \text{class}\end{smallmatrix} \right|$$

$$\Downarrow$$

$$|G| \overset{O\text{-}S}{=} \sum_{\text{conj. classes}} \frac{|G|}{|C_G(h)|}$$

this equality is called the class equation of G

$$\text{equivalently, } 1 = \sum_{\text{conj. classes}} \frac{1}{|G_G(h)|}$$

some representative of each conj. class

Example:

$$D_6 = \underset{\text{orbit of size } 1}{\{e\}} \sqcup \overset{\text{rotations}}{\underset{\text{size } 2}{\{r, r^2\}}} \sqcup \overset{\text{reflections}}{\underset{\text{size } 3}{\{s_1, s_2, s_3\}}}$$

Class equation

$$6 = 1^{=\frac{6}{6}} + 2^{=\frac{6}{3}} + 3^{=\frac{6}{2}}$$

$$= \frac{6}{\underbrace{|C_{D_6}(e)|}_{6}} + \frac{6}{\underbrace{|C_{D_6}(\pi)|}_{3}} + \frac{6}{\underbrace{|C_{D_6}(s)|}_{2}}$$